Studies of the Vector-Meson Mass Generation Scheme by Chiral Anomalies in two-Dimensional non-Abelian Gauge Theories

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Abstract

Higher order effects of the two-dimensional non-Abelian gauge theories, in which the vector-meson mass is generated by chiral anomalies, will be studied. The β function and the topological nature of the non-linear σ model in the action and unitarity of the theory will be discussed.

Introduction$-$ The fact that chiral symmetry in a Lagrangian at tree level does not survive quantization is well-established, and widely-studied[1], althought we still cannot answer the question whether these anomalies have any physical significance. There are various ways to remove chiral anomaly[2]. Some leads to quite successful prediction[2]. In Jackiw and Rajaraman's work[3], they adopted an alternative approach. They considered the two-dimensional chiral Schwinger model, and, by simply giving up gauge invariance, obtained a consistent and unitary theory, in which the vector meson necessarily acquires a mass when consistency and unitarity are demanded.

R. Rajaraman extended the work to non-Abelian case[6]. At tree level, the non-Abelian chiral Schwinger model is invariant under gauge transformations G, where we can take G as an $SU(N)$ matix without loss of generality. However, the chiral couplings between the gauge fields and fermionic fields will introduce an anomaly. By using the non-Abelian bosonization scheme[4], the anomaly will appear in the bosonized action at its tree level, and the bosonized form of the model is

$$S_B = \sum_{i=1}^{n} S[U_i] + \int d^2x tr[\frac{1}{2g^2}F^2 + \frac{i}{2\pi}A_- \sum_{i=1}^{n} U_i^\dagger \partial_+ U_i + \frac{n\alpha}{4\pi}A_+A_-], \tag{1}$$

where the gauge fields $A^\pm = A_\mp = (1/\sqrt{2})(A^0 \pm A^1)$, and $\partial_\pm = (1/\sqrt{2})(\partial_0 \pm \partial_1)$, and $S[U_i]$ is a Wess-Zumino-Witten action describing a conformally invariant nonlinear σ model which is equivalent to one flavor of N free massless Dirac fermions in two dimensions

$$S[U] = \frac{1}{4\pi}\int d^2x tr\partial_+ U \partial_- U^\dagger + \frac{1}{12\pi}\int_B d^3y \epsilon^{ijk} trU^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U. \tag{2}$$

The second term in (2) is the Wess-Zumino term, and is defined as an integral over a three-dimensional manifold B whose boundary is two-dimensional compactified space-time.

For the gauge field, the gauge transformation for $A^a_\pm$ is $A^G_\pm = GA_\pm G^+ + i(\partial_\pm G)G^+$, and for U_i, from the bosonization scheme, it is given by $U_i \longrightarrow U_i G^+$.

The last term in (1), a mass term for the gauge field with an arbitrary coefficient α, reflects an ambiguity in computing the fermionic determinant for a given gauge field configuration. The anomaly can not be identically zero for any choice of α[6].

Following Faddeev's suggestion[6], one can introduce an extra field V in eq. 1 to have

$$S_B = \sum_{i=1}^n S[U_i] + nS[V^\dagger] + \int d^2x tr\{\tfrac{1}{2g^2}F^2 + \tfrac{i}{2\pi}A_- \sum_{i=1}^n U_i^\dagger \partial_+ U_i +$$

$$\tfrac{n}{4\pi}[\alpha A_+ A_- + (\alpha - 2)\partial_+ V^\dagger \partial_- V + i\alpha A_+ V^\dagger \partial_- V + i(\alpha - 2)A_- V^\dagger \partial_+ V]\}.$$

This action is, by design, gauge invariant under $A_\pm \longrightarrow A^G_\pm$, $U_i \longrightarrow U_i G^\dagger$, and $V^\dagger \longrightarrow GV^\dagger$. In other words, the newly added current couples to the gauge fields with opposite chirality from the original current. Therefore the anomalies from the newly added scalar fields V cancel those of the scalar fields U. The gauge choice $V = 1$ reduces it to the original system (1), i.e. the new equation is equivalent to eq.(1) and no new degrees of freedom are introduced as compared to the system in eq.(1).

From ref.[6], we also know that (1) determines a sensible theory, even though it is not gauge invariant, provided α exceeds 1[6,7]. Contrary to a gauge theory in 1+1 dimensions, which is a trivial one, because of the anomaly, it has been shown[6,7,8] that with the regularization $\alpha > 1$, the space components A^a_1 of the vector-field multiplet (and their canonical momenta E^a) survive as dynamical variables, in addition to the matter fields.

However the consistence we talked about is only at tree-level (or at one-loop level for the fermionic theory since the bosonized action (1) contains fermionic radiative effects and the coefficient α represents regularization ambiguities). Eq.(1) is not completely solvable. We don't know whether it is still a consistent theory, e.g., unitary and hermitian, etc., if perturbation theory is employed. Furthermore we don't know whether the theory is renormalizable. In this paper, I will look into these matters.

Quantization$-$In the following, I consider the gauge group $\mathcal{G} = SU(2)$. But the method is general.

By considering the constraints in (1) as second class and replacing all the Poission brackets by Dirac brackets[10], R. Rajaraman[6,8] has discussed the quantization of the theory. However, from eq.(1), we know that the canonical dimension for the vector field operator is $d(A) = 1$. Then by power counting, the renormalizability of (1) is not straight forward. For the convenience of discussion, I will adopt Faddeev's proposal[11] to quantize the theory. Introducing an extra field V, where $V = \sqrt{2}(\Psi^0\lambda^0 + i\Psi^a\lambda^a)$ is also an element in $SU(2)$ group with $\Psi^0\Psi^0 + \Psi^a\Psi^a = 1, \lambda^0 = 1/\sqrt{2}, Tr(\lambda^a\lambda^b) = \delta^{ab}, [\lambda^a, \lambda^b] = if^{abc}\lambda^c =$

$i\sqrt{2}\epsilon^{abc}\lambda^c$, the enlarged gauge invariant action of eq.(5) will be[7,12]

$$S_B = \sum_{i=1}^{n}\{(1/4\pi)\int d^2x\, g_{ab}[\vec{\Phi}_i]\partial_\mu\Phi_i^a\partial^\mu\Phi_i^b - (1/6\pi)\int d^2x\,\epsilon^{\mu\nu}e_{ab}[\vec{\Phi}_i]\partial_\mu\Phi_i^a\partial_\nu\Phi_i^b\} +$$

$$n\{[(\alpha-1)/4\pi]\int d^2x\, g_{ab}[\vec{\Psi}]\partial_\mu\Psi^a\partial^\mu\Psi^b + (1/6\pi)\int d^2x\,\epsilon^{\mu\nu}e_{ab}[\vec{\Psi}]\partial_\mu\Psi^a\partial_\nu\Psi^b\} +$$

$$\int d^2x\,\{-(1/4)F^a_{\mu\nu}F^{a\mu\nu} + (n\alpha g^2/8\pi)A^a_\mu A^{a\mu} -$$

$$(\sqrt{2}g/4\pi)(g^{\mu\nu}+\epsilon^{\mu\nu})\sum_{i=1}^{n}A^a_\mu[\Phi_i^0\partial_\nu\Phi_i^a - \Phi_i^a\partial_\nu\Phi_i^0 + \epsilon^{abc}\Phi_i^b\partial_\nu\Phi_i^c] -$$

$$(\sqrt{2}gn/4\pi)[(\alpha-1)g^{\mu\nu} - \epsilon^{\mu\nu}]A^a_\mu[\Psi^0\partial_\nu\Psi^a - \Psi^a\partial_\nu\Psi^0 + \epsilon^{abc}\Psi^b\partial_\nu\Psi^c]\}, \quad (3)$$

with[9]

$$g_{ab}[\vec{\Phi}] = \delta_{ab} + (1-\vec{\Phi}^2)^{-1}\Phi_a\Phi_b, \quad (4)$$

$$e_{ab}[\vec{\Phi}] = \epsilon^{abc}\frac{\pm 3\Phi^c}{2|\vec{\Phi}|^3}[\arcsin|\vec{\Phi}| - (\vec{\Phi}^2 - \vec{\Phi}^4)^{1/2}], \quad (5)$$

where we have let $A^a_\mu \longrightarrow gA^a_\mu$, $U = \sqrt{2}(\Phi^0\lambda^0 + i\Phi^a\lambda^a)$. The action is clearly local gauge invariant under $A_\pm \longrightarrow A^G_\pm$, $U_i \longrightarrow U_iG^\dagger$, and $V^\dagger \longrightarrow GV^\dagger$. We see that, upon using the ghost-free gauge $V^\dagger = 1$, one restores the original system in eq.(1), and eq.(1) clearly contains the same degrees of freedom for the enlarged system (3). Therefore, the action (1) is just the action (3) in the gauge $V^\dagger = 1$ at tree level.

Considering the symmetry of the theory and the simplification of the discussion, as well as the transparence of renormalizability, I will choose the covariant, or Lorentz, gauge to fix the gauge invariance

$$\partial^\mu A^a_\mu = 0. \quad (6)$$

This gauge condition is not ghost-free. The generating functional for the theory will be

$$Z_B[\vec{J},\vec{K},\vec{L},\vec{\alpha},\vec{\beta}] = \int[d\vec{A}_\mu](\prod_{i=1}^{n}[\frac{d\vec{\Phi}_i}{\sqrt{1-\vec{\Phi}^2}}])[\frac{d\vec{\Psi}}{\sqrt{1-\vec{\Psi}^2}}][d\vec{c}][d\vec{c^+}]exp\, i\int d^2x\{\mathcal{L}_{eff} +$$

$$J^a_\mu A^{a\mu} + \sum_{i=1}^{n}K_i^a\Phi_i^a + L^a\Psi^a + c^{a+}\alpha^a + \beta^a c^a\}, \quad (7)$$

where

$$S_{eff} = \int d^2x \mathcal{L}_{eff} = S_B - \int d^2x \{(1/2\xi)(\partial^\mu A^a_\mu)^2 + c^\dagger_a \partial^\mu [\delta_{ab}\partial_\mu - g f_{abc} A^c_\mu] c_b(x)\}. \qquad (8)$$

In eq.(8), S_B is given in eq.(3), $-(1/2\xi)(\partial^\mu A^a_\mu)^2$ is the gauge-fixing term, and c^a is the Faddeev-Popov(FP) ghost coming from the gauge fixing condition, and we have introduced the sources J^a_μ, K^a, L^a, α^a, and β^a for the fields A^a_μ, Φ^a, Ψ^a, c^{a+}, and c^a respectively.

As we can see from the quadratic terms of the effective Lagrangian, there are mixings among Φ^a, Ψ^a, and A^a_μ. From the quadratic term in eqs.(3,7), we can obtain the propagators for $(\Phi^a, \Psi^a, A^a_\mu)$ as 3×3 matrix G^{ac}_{ij} and the FP ghost propagator. Getting the form of the vertices from the interaction terms of the effective Lagrangian (8)is straight forward.

Detailed calculations can clearly show that the canonical dimension for the scalar fields Φ^a, Ψ^a and the gauge fields A^a_μ are zero in the covariant gauge,in which the theory has good high-energy behaviour, therefore the theory is renomalizable. However, we also require the renormalized Lagrangian to be gauge-invariant and to have the same degrees of freedom in all orders, such that we can be sure the equivalence of the theory in different gauges and the renormalizability, unitarity, and Lorentz invariance of the action given in (1).

Regularization and Renormalization$-$ Since we can prove that all the one-loop momentum integrals involving odd number of ϵ-tensors are finite or can be considered as connected with external lines (see ref.[13] for details), we can safely use dimensional regularization at one-loop level, which does not destroy the gauge invariance. For two-loop (or higher-order calculations), we can discuss them after we obtain the one-loop (or corresponding lower-order) renormalized lagrangian (also see ref.[13] for details). The additional interaction of the form $-(1/2)\delta^2(0)\int d^2x ln[1 - \vec{\Phi}(x)^2]$ from the interaction measure due to the constraint $(\Phi^0)^2 + (\vec{\Phi}^a)^2 = 1$ will not affect our discussion[13]. In order to avoid the singularities due to the scalar propagators below two dimensions for the massless scalar theory and for simplicity of the Ward$-$Takahashi(WT) identities coming later, we choose the infrared cutoff as

$$\int d^2x[(\pm)H(1 - \vec{\Phi}^2)^{1/2} + (\pm)I(1 - \vec{\Psi}^2)^{1/2}], \qquad (9)$$

where H and I are arbitrary sources coupled to Φ^0 and Ψ^0. Indeed the expansions of (12) in powers of $\vec{\Phi}^2(x)$ and $\vec{\Psi}^2(x)$ generate masses for the scalar particles. Eventually, H and I will need to take the limit zero.

I shall now assume that the lagrangian has been regularized and discuss the WT identities and their implications for the structure of the counterterm. Without loss of generality, I will assume, in the following, that there is only one flavour (i.e. $n = 1$) in the theory.

The gauge-fixed effective action (8) is obviously not invariant under the general gauge transformation $G = 1 + i\Omega^a\lambda^a$ with an arbitrary Ω^a, but it is invariant under the transformation with $\Omega^a = -gc^a\delta\lambda$, where $\delta\lambda$ is a Grassmann number.

Introducing the source terms Q^a_μ, ν^a, δ^0, δ^a, η^0, η^a for the composite operators $(D^\mu\vec{c})^a$, $(g/2)(\vec{c}\times\vec{c})^a$, $(g\vec{\lambda}\cdot\vec{c}U^\dagger)^0$, $(g\vec{\lambda}\cdot\vec{c}U^\dagger)^a$, $(g\vec{\lambda}\cdot\vec{c}V^\dagger)^0$, $(g\vec{\lambda}\cdot\vec{c}V^\dagger)^a$ respectively, the generating functional will be of the form

$$
\begin{aligned}
Z_B[\vec{J},\vec{K},\vec{L},\vec{\alpha},\vec{\beta},Q^a_\mu,\nu^a,\delta^0,\delta^a,\eta^0,\eta^a] &= \int\prod_x[d\vec{A}_\mu][\frac{d\vec{\Phi}}{(1-\vec{\Phi}^2)^{1/2}}][\frac{d\vec{\Psi}}{(1-\vec{\Psi}^2)^{1/2}}][d\vec{c}][d\vec{c}^\dagger] \\
& \qquad exp\, i\int d^2x\{\mathcal{L}_{eff}+\Sigma\}, \qquad (10)
\end{aligned}
$$

where S_{eff} is given in eq.(8) and

$$
\begin{aligned}
\Sigma &= J^a_\mu A^{a\mu} + K^a\Phi^a + L^a\Psi^a + c^{a\dagger}\alpha^a + \beta^a c^a + \delta^0(g\vec{\lambda}\cdot\vec{c}U^\dagger)^0 + \delta^a(g\vec{\lambda}\cdot\vec{c}U^\dagger)^a + \\
& \quad \eta^0(g\vec{\lambda}\cdot\vec{c}V^\dagger)^0 + \eta^a(g\vec{\lambda}\cdot\vec{c}V^\dagger)^a + Q^a_\mu(D^\mu\vec{c})^a + \nu^a(g/2)(\vec{c}\times\vec{c})^a - \\
& \quad (H/2\pi)(1-\vec{\Phi}^2)^{1/2} - (I/2\pi)(1-\vec{\Psi}^2)^{1/2}. \qquad (11)
\end{aligned}
$$

From $\delta Z = 0$ under the transformation $G = 1 + i(-gc^a\delta\lambda)\lambda^a$, it is not difficult to show

$$
\begin{aligned}
& \delta\lambda\int d^2x[-\frac{gH}{2\sqrt{2}\pi i}\frac{\delta}{i\delta\delta^0} - \frac{gI}{2\sqrt{2}\pi i}\frac{\delta}{i\delta\eta^0} - J^a_\mu\frac{\delta}{i\delta Q^a_\mu} + \frac{1}{\sqrt{2}}K^a\frac{\delta}{i\delta\delta^a} + \frac{1}{\sqrt{2}}L^a\frac{\delta}{i\delta\eta^a} + \\
& \frac{i}{\xi}\alpha^a\partial_\mu\frac{\delta}{i\delta J^a_\mu} - \beta^a\frac{\delta}{i\delta\nu^a}]\ exp\ i\int d^2y[\mathcal{L}_{eff}+\Sigma] = 0. \qquad (12)
\end{aligned}
$$

Eq.(11) is the generalized WT identity which relates different types of Green's functions.

The generating functional of the connected Green's functions $W[\vec{J},\vec{K},\vec{L},\vec{\alpha},\vec{\beta},Q^a_\mu,\nu^a,$

$\delta^0, \delta^a, \eta^0, \eta^a] = i\, lnZ$ of the fields $\vec{\Phi}$, $\vec{\Psi}$, $\vec{A}_\mu$, satisfies the same equation as (11). Defining

$$
\begin{aligned}
-\Gamma[\vec{a}_\mu, \vec{\phi}, \vec{\psi}, \vec{C}^\dagger, \vec{C}, Q^a_\mu, \nu^a, \delta^0, \delta^a, \eta^0, \eta^a] \quad &= \quad \int d^2x[a^a_\mu J^{a\mu}(x) + \phi^a K^a + \psi^a L^a + \\
&\qquad C^{\dagger a}\alpha^a + \beta^a C^a] + W, \qquad (13)
\end{aligned}
$$

where

$$
\begin{array}{rcll}
a^a_\mu &=& -\delta W/\delta J^{a\mu}, & J^{a\mu} = -\delta\Gamma/\delta a^a_\mu, \\
\phi^a &=& -\delta W/\delta K^a, & K^a = -\delta\Gamma/\delta\phi^a, \\
\psi^a &=& -\delta W/\delta L^a, & L^a = -\delta\Gamma/\delta\psi^a, \\
C^{\dagger a} &=& \delta W/\delta\alpha^a, & \alpha^a = -\delta\Gamma/\delta C^{a\dagger}, \\
C^a &=& -\delta W/\delta\beta^a, & \beta^a = \delta\Gamma/\delta C^a,
\end{array} \qquad (14)
$$

and setting

$$
\Gamma = \tilde{\Gamma} - \frac{i}{2\xi}\int d^2x(\partial_\mu a^{a\mu})(\partial_\nu a^{a\nu}), \qquad (15)
$$

then we have the equations for $\tilde{\Gamma}$ from eq.(11) as

$$
\begin{aligned}
\int d^2x[\quad & \frac{H}{2\sqrt{2}\pi}\frac{\delta\tilde{\Gamma}}{\delta\delta^0} + \frac{I}{2\sqrt{2}\pi}\frac{\delta\tilde{\Gamma}}{\delta\eta^0} + i\frac{\delta\tilde{\Gamma}}{\delta a^{a\mu}}\frac{\delta\tilde{\Gamma}}{\delta Q^a_\mu} + \frac{i}{\sqrt{2}}\frac{\delta\tilde{\Gamma}}{\delta\phi^a}\frac{\delta\tilde{\Gamma}}{\delta\delta^a} + \frac{i}{\sqrt{2}}\frac{\delta\tilde{\Gamma}}{\delta\psi^a}\frac{\delta\tilde{\Gamma}}{\delta\eta^a} + \\
& i\frac{\delta\tilde{\Gamma}}{\delta C^a}\frac{\delta\tilde{\Gamma}}{\delta\nu^a}] = 0, \qquad (16)
\end{aligned}
$$

$$
\frac{\delta\tilde{\Gamma}}{\delta C^{a\dagger}} + \partial_\mu\frac{\delta\tilde{\Gamma}}{\delta Q^a_\mu} = 0. \qquad (17)
$$

Performing a loop expansion of the functional $\tilde{\Gamma}$, the corresponding Feynman diagrams are obtained by expanding the action and the integration measure in powers of $\vec{\Phi}^2$, $\vec{\Psi}^2$ to the appropriate order. At lowest order, it is not hard to show that $\tilde{\Gamma}$ is given by

$$
\begin{aligned}
\tilde{\Gamma}^{(0)} \quad &= \quad S_B + \int d^x\{ic^\dagger_a(x)\partial^\mu[\delta_{ab}\partial_\mu - gf_{abc}A^c_\mu]c_b - (H/2\pi)(1-\vec{\phi}^2)^{1/2} - \\
&\qquad (I/2\pi)(1-\vec{\psi}^2)^{1/2} + Q^a_\mu(D^\mu\vec{c})^a + \nu^a\frac{g}{2}(\vec{c}\times\vec{c})^a + \delta^0(g\vec{\lambda}\cdot\vec{c}U^\dagger)^0 + \\
&\qquad \delta^a(g\vec{\lambda}\cdot\vec{c}U^\dagger)^a + \eta^0(g\vec{\lambda}\cdot\vec{c}V^\dagger)^0 + \eta^a(g\vec{\lambda}\cdot\vec{c}V^\dagger)^a\}. \qquad (18)
\end{aligned}
$$

For one-loop order, when the cutoff increases, the divergent part of $\Gamma^{(1)}$ is singled out. By adding to the action a counterterm $t \cdot [-\tilde{\Gamma}^{(1)div} + O(t)]$, then a renormalized action $(S + tS_1)$ can be constructed and satisfies the transformation invariance given above.

Noting that $\tilde{\Gamma}^{(1)div}$ is a local function of dimension 2 of the $\vec{\Phi}$, $\vec{\Psi}$, $\vec{A_\mu}$ fields, and $H(x)$, $I(x)$ are also of dimension 2, by power counting, we can solve the WT identities and write the rescaled action in terms of the rescaled fields as

$$
\begin{aligned}
S_B^R \;=\; & \int d^2x\{(Z_1/4\pi Z_2)g_{ab}^R[\vec{\phi}]\partial_\mu\phi^a\partial^\mu\phi^b - (Z_1/6\pi Z_3)\epsilon^{\mu\nu}e_R^{ab}(\vec{\phi})\partial_\mu\phi^a\partial_\nu\phi^b] + \\
& [(\alpha^R - 1)Z_4/4\pi Z_5]g_{ab}^R\ [\vec{\psi}]\partial_\mu\psi^a\partial^\mu\psi^a + (Z_4/6\pi Z_6)\epsilon^{\mu\nu}e_R^{ab}(\vec{\psi})\partial_\mu\psi^a\partial_\nu\psi^b - \\
& \frac{1}{4}[\partial_\mu a_\nu^{Rb} - \partial_\nu a_\mu^{Rb} - g^R f^{abc}a_\mu^{Rb}a_\nu^{Rc}][\partial^\mu a^{Rb\nu} - \partial^\nu a^{Rb\mu} - g^R f^{bmn}a^{Rm\mu}a^{Rn\nu}] + \\
& \frac{\alpha^R(g^R)^2}{8\pi}a_\mu^{Rb}a^{Rb\mu} - \frac{\sqrt{2}}{4\pi}(\frac{Z_1}{Z_2}g^R g^{\mu\nu} + \frac{Z_1}{Z_3}g^R\epsilon^{\mu\nu})a_\mu^{Ra}[\pm(\frac{1}{Z_1} - \vec{\phi}^2)^{1/2}\partial_\nu\phi^a + \\
& (\pm)\phi^a\frac{\phi^m\partial_\nu\phi^m}{((1/Z_1) - \vec{\phi}^2)^{1/2}} + \epsilon^{abc}\phi^b\partial_\nu\phi^c] - \frac{\sqrt{2}}{4\pi}[(\alpha^R - 1)\frac{Z_4}{Z_5}g^R g^{\mu\nu} - \frac{Z_4}{Z_6}g\epsilon^{\mu\nu}] \times \\
& a_\mu^{Ra}[\pm(\frac{1}{Z_4} - \vec{\psi}^2)^{1/2}\partial_\nu\psi^a + (\pm)\psi^a\frac{\psi^m\partial_\nu\psi^m}{((1/Z_4) - \vec{\psi}^2)^{1/2}} + \epsilon^{abc}\psi^b\partial_\nu\psi^c]\,\} - \\
& (H/2\pi)[(1/Z_1) - \vec{\phi}^2]^{1/2} - (I/2\pi)[(1/Z_4) - \vec{\psi}^2]^{1/2}, \qquad (19)
\end{aligned}
$$

and similarly for S_{FPG}^R, S_{GF}^R, Q_μ^a, ν^a, etc., where

$$
\begin{aligned}
& Z_1 = 1 + 2tB_1, \\
& Z_1/Z_2 = 1 - t[\iota - 2\kappa - 2B_1 - (4\pi/\sqrt{2}g)B_2], \\
& Z_1/Z_3 = 1 - t[\iota - 2\kappa - 2B_1 - (4\pi/\sqrt{2}g)B_5], \\
& Z_4 = 1 + 2tB_3, \\
& \frac{Z_4}{Z_5} = 1 + t[\frac{\iota - 2\kappa}{\alpha - 1} + 2B_3 - \frac{4\pi}{\sqrt{2}g}\frac{B_2}{\alpha - 1}], \\
& Z_4/Z_6 = 1 - t[\iota - 2\kappa - 2B_3 - (4\pi/\sqrt{2}g)B_5], \\
& a_\mu^{Ra} = {Z_7}^{1/2}a_\mu^a = [1 + t(\kappa - \iota)]a_\mu^a, \\
& g^R = Z_8{Z_7}^{-3/2}g = [1 - 2t(\kappa - \iota)]g, \\
& Z_9 = 1 - (t\,\iota/2),
\end{aligned}
$$

$$\alpha^R = \alpha + t[\alpha(2\kappa - \iota) + (4\pi/\sqrt{2}g)(B_2 + B_4)],$$

$$g^R_{ab}(\vec{\phi}) = \delta_{ab} + [(1/Z_1) - \vec{\phi}^2]^{-1}\phi^a\phi^b,$$

$$e^{ab}_R = \epsilon^{abc}\frac{\pm 3\phi^c}{2|\vec{\phi}|^3}[\arcsin|\vec{\phi}| - |\vec{\phi}|(1/Z_1 - \vec{\phi}^2)^{1/2}], \tag{20}$$

and we have assumed the facts of unbroken global gauge symmetry and ghost number conservation. Therefore we achieve the renormalization at one-loop order.

After this renormalization, the gauge invariance of S_B remains, but the transformation law of the fields is performed under the rescaled fields and coupling constants

$$U^R = \pm(1 - Z_1\vec{\Phi}^2)^{1/2}\lambda^0 + iZ_1{}^{1/2}\Phi^a\lambda^a \longrightarrow U^R G^\dagger,$$

$$V^R = \pm(1 - Z_3\vec{\Psi}^2)^{1/2}\lambda^0 + iZ_3{}^{1/2}\Psi^a\lambda^a \longrightarrow V^R G^\dagger,$$

$$(A^R_\mu/g^R)^G \longrightarrow G(A^R_\mu/g^R)G^+ + i(\partial_\mu G)G^+, \tag{21}$$

with G in the same group representation as the one given before, and the integration measure should be modified by $d\vec{\Phi}/[(1/Z_1) - \vec{\Phi}^2]^{1/2}$ and $d\vec{\Psi}/[(1/Z_4) - \vec{\Psi}^2]^{1/2}$. However these modifications of the interaction will only affect the two-loop order. Thus the renormalized lagrangian is gauge invariant, and the gauge group and its representation remains the same.

For higher loop situation, we can proceed the process inductively. Assuming that a renormalized action up to order $(n-1)$, which satisfied the WT identities, has been constructed, then we can construct the loop expansion up to order n with this action. Taking the large cutoff limit, we will realize that $\tilde{\Gamma}^{(n)div}$ satisfies the same equation as $\tilde{\Gamma}^{(1)div}$. The integration of the equation for $\tilde{\Gamma}^{(n)div}$ has already been discussed, and we see that, at order n, the effects of the renormalization may be again absorbed into rescaling of the fields and the coupling constants.

This completes the induction and shows that for this parametrization of the system, the theory is renormalizable and the renormalized action is gauge invariant, and the gauge group, its representation, and the degrees of freedom remain the same. The renormalized action has the form given by eq.(19). The above discussions also leads to the proof of the renormalizability of the action (1) and the equivalence between eq.1 and eq.3 at all orders.

More Discussions$-$ The B_i's appearing in eq.(20) can be obtained through perturbative calculation. Renormalization group behaviour and topological nature of the non-linear σ model in two dimensions has been well studied[14]. In the case at hand, due to scalar-vector couplings, the affection of these couplings to the β function and to the topological nature should be re-examined.

The contribution from the gauge field can come from the propagators and the scalar-vector couplings. The Feynman diagrams that could affect the calculation of the β function of the non-linear σ model are two- and four-point diagrams of the scalar fields. For the diagrams with explicit scalar-vector coupling, since the counter terms for them doesn't exist in the tree level Lagrangian, the renormalizability of the model would require the sum of them to be finite. For the diagrams without explicit scalar-vector coupling, detailed calculations show that the contributions from the gauge fields through the propagators are finite. Therefore, at least at one-loop order, the scalar-gauge field couplings do not affect the β function of the model. The β function behavior of the scalar field interactions, e.g. $\vec{\Phi}$ or $\vec{\Psi}$, will be the same as the pure scalar field theories, depending on the coefficients in the WZW action.

Detailed calculations can show that $Z_1/Z_3 = Z_4/Z_6 = 1$, which guarantees the topological nature of the nonlinear σ model.

From eq.18, we know that the renormalized action has the same form as eq.3. Therefore gauge invariance and the discussion in [6,7,8] guarantee the unitarity of eq.1 in higher order, provided $\alpha^R > 1$.

Up to this point, higher order effects and gauge invariance of the theory have been studied, and the β function and the topoplogical nature of the nonlinear σ model, as well as the untarity of the theory have been re-examined. More properties should be further studied. Hope this work can be one step closer than using the tree-level action to draw conclusion.

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